

# Reversibility of local transformations of multiparticle entanglement

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(07 December 1999)

We consider the transformation of multisystem entangled states by local quantum operations and classical communication. We show that, for any reversible transformation, the relative entropy of entanglement for any two parties must remain constant. This shows, for example, that it is not possible to convert  $2N$  three-party GHZ states into  $3N$  singlets, even in an asymptotic sense. Thus there is true three-party non-locality (i.e. not all three party entanglement is equivalent to two-party entanglement). Our results also allow us to make *quantitative* statements about concentrating multi-particle entanglement. Finally, we show that there is true  $n$ -party entanglement for any  $n$ .

One of the key open issues in quantum information theory is to understand what the fundamentally different types of quantum entanglement are. It has been known for some time [1] that any pure entangled state of two parties, Alice and Bob, may be reversibly distilled to singlets in the sense that in the limit of large  $N$ ,  $N$  copies of the state  $\psi^{AB}$  may be transformed reversibly into  $NE(\psi^{AB})$  singlets, where  $E(\psi^{AB})$  is the entropy of the reduced density matrix of either Alice or Bob. Until recently it was not known whether, in fact, singlets are the only type of entanglement. This issue was resolved in [2] where was shown that certain multi-party states cannot be transformed into singlets reversibly. In particular the authors of [2] consider the four-party GHZ state:

$$\psi_{GHZ_4} = \frac{1}{\sqrt{2}} (|0000\rangle + |1111\rangle). \quad (1)$$

Let us imagine that there were a reversible process to convert  $N$  copies of  $\psi_{GHZ_4}$  into singlets (in the limit of large  $N$ ). After the forward version of the process there would be  $s^{AB}$  singlets between Alice and Bob,  $s^{BC}$  singlets between Bob and Claire etc. It is straightforward to calculate the four one-party entropies (e.g. the entropy of Alice versus the Bob-Claire-Daniel system) and three independent two-party entropies (e.g. the entropy of the Alice-Bob system versus the Claire-Daniel system) for both the initial and final pure states.

Since entropy can only decrease during any local process, it must be constant during a reversible process. Thus a *necessary* condition for the existence of a reversible protocol for converting  $N$  copies of  $\psi_{GHZ_4}$  into singlets is that the entropies of the initial and final states must be the same. It is not difficult to show that no combinations of singlets held between the four parties has the same ratios of entropies as  $\psi_{GHZ_4}$ . Thus the four-party GHZ state cannot be converted reversibly into singlets. Hence not all four-party entanglement is of the singlet

type.

[2] leaves open many questions. For example, while the results in [2] show that not all four-party entanglement can be attributed to pair-wise entanglement, the techniques employed leave open whether or not any new types of non-locality arise in three-party states. Consider the three one-party entropies,  $E^A$ ,  $E^B$ ,  $E^C$  of an arbitrary three party pure state. It can easily be checked that any values of these three entropies which are allowed by sub-additivity can be matched by suitable choices of the numbers of singlets held between the three parties.

A particularly important case is that of the three party GHZ state

$$\psi_{GHZ_3} = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle). \quad (2)$$

A simple calculation shows that the one-party entropies of  $\psi_{GHZ_3}$  can be matched by having one singlet between each pair  $AB$ ,  $BC$ ,  $AC$  for every two  $\psi_{GHZ_3}$ 's held between the parties. There is further encouragement for the suggestion that, in fact, the three party GHZ state is equivalent to singlets, since it is possible to create singlets between any pair from  $\psi_{GHZ_3}$ . To see this rewrite (2) as

$$\psi_{GHZ_3} = \frac{1}{\sqrt{2}} \left[ \frac{(|0\rangle + |1\rangle)(|00\rangle + |11\rangle)}{\sqrt{2}} + \frac{(|0\rangle - |1\rangle)(|00\rangle - |11\rangle)}{\sqrt{2}} \right]. \quad (3)$$

Now let Alice measure her particle in the  $x$ -basis,

$$|+\rangle_x = \frac{(|0\rangle + |1\rangle)}{\sqrt{2}}; \quad |-\rangle_x = \frac{(|0\rangle - |1\rangle)}{\sqrt{2}}. \quad (4)$$

If she finds her particle in the  $+x$  direction she tells Bob and Claire to do nothing, if she finds her particle in the

$-x$  direction she tells Bob to do the unitary transformation on his particle (a rotation about the  $z$ -axis)

$$|0\rangle \mapsto |0\rangle; \quad |1\rangle \mapsto -|1\rangle \quad (5)$$

At the end of these operations Bob and Claire share the state,

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \quad (6)$$

Thus given one three party GHZ state, one singlet between Bob and Claire can be produced. However the only known protocol for converting singlets to GHZ states consumes two singlets; for example given singlets between Alice and Bob and between Bob and Claire, Bob could create a GHZ state locally and then use the singlets to teleport to Alice and Claire. Thus while individual copies of GHZ states can be converted to singlets and vice versa, the only known protocol is not reversible. Indeed it has been shown that no reversible protocol exists for finite numbers of copies [2]. A natural question is whether in the asymptotic limit any such reversible protocol can exist. Our results below settle the question.

In this letter we first investigate the three-party case. We will consider general collective actions on any number,  $N$ , of copies of the state (including the asymptotic limit  $N \rightarrow \infty$ ). We derive new conditions that any procedure involving local quantum operations and classical communication must satisfy, namely that the increase in the average relative entropy of entanglement of any pair of the three parties cannot be greater than the decrease in the entanglement of the third with the pair. A corollary of this result is that the average relative entropy of entanglement of any pair of the parties must be constant in a reversible process. This allows us easily to conclude that not all three party entanglement is of singlet type since we can show that there is no reversible process converting the three party GHZ state into singlets, even in the asymptotic limit. More generally we will show that there is true  $n$ -party entanglement for any  $n$ , in the sense that, for any  $n$  there are  $n$ -party states which cannot be transformed reversibly into states in which only  $k < n$  parties are entangled.

It will simplify matters if we consider the entire Hilbert space at each site including ancilla's. Thus initially Alice, Bob and Claire share a pure state. They are allowed to perform local operations and classical communication, as usual. The state of the system branches according to the outcome of any measurements performed, but at each stage the state in a given branch is still pure. The set of local operations can be broken down into two classes:

- Alice Bob and Claire can do local unitary operations on their systems, and since they can communicate, everybody can know what transformations are done by everybody else. Local unitary transformations cannot increase the entanglement.
- They can perform more general operations, including measurements. Communication means that everybody can know the results of every measurement performed by anybody.

We note that entanglement between Bob and Claire can be increased. For example, a measurement performed by Alice can increase the entanglement of Bob and Claire. The example of the three-party GHZ state above shows this since by taking the trace over Alice's Hilbert space one can see that Bob and Claire's relative state is unentangled in the case of the three-party GHZ state. However after the protocol Bob and Claire's entanglement is one e-bit. It is relevant, however (see below) that after the protocol Alice is unentangled with Bob and Claire.

Our aim below is to calculate how much Bob and Claire's entanglement can increase under the most general local operations that the three parties can perform.

Specifically we will show that any increase in the Bob-Claire bipartite entanglement must be paid for by a decrease in the entanglement of Alice with the other two—that is (since we always have a pure state) by a decrease in Alice's entropy.

The proof is as follows. Consider an entanglement manipulation protocol which states with any number of copies of a three-party entangled state. Consider a particular stage in the protocol and a particular branch in which the density matrix of the system is  $\rho^{ABC}$  (note that the state is pure, and lives in the Hilbert space of all the original copies including the ancillas); the state of Bob and Claire's joint system is  $\rho^{BC} = \text{Tr}_A(\rho^{ABC})$ . Let Alice perform a measurement of an operator with spectral projectors  $P_k$ . Thus if the outcome  $k$  is obtained, the state of the system is

$$\frac{P_k \otimes \mathbf{I} \otimes \mathbf{I} \rho^{ABC} P_k \otimes \mathbf{I} \otimes \mathbf{I}}{\text{Tr}(P_k \otimes \mathbf{I} \otimes \mathbf{I} \rho^{ABC})}, \quad (7)$$

where  $p_k = \text{Tr}(P_k \otimes \mathbf{I} \otimes \mathbf{I} \rho^{ABC})$  is the probability that the outcome  $k$  is obtained. We note that

$$\rho^{BC} = \sum_k p_k \rho_k^{BC}, \quad (8)$$

where  $\rho_k^{BC}$  is the state of Bob and Claire's joint system after the measurement; in other words after Alice's measurement, but before she has communicated the outcome to Bob and Claire, their average state cannot have changed from what it was before the measurement.

The relative entropy of  $\rho^{BC}$  with respect to any bipartite state  $\sigma^{BC}$  is defined as

$$S(\rho^{BC} || \sigma^{BC}) := \text{Tr}(\rho^{BC} \ln \rho^{BC}) - \text{Tr}(\rho^{BC} \ln \sigma^{BC}) \quad (9)$$

Simple algebra shows that the relative entropy satisfies "Donald's identity" [3],

$$\begin{aligned} & -S(\rho^{BC}||\sigma^{BC}) \\ & = \sum_k p_k (S(\rho_k^{BC}||\rho^{BC}) - S(\rho_k^{BC}||\sigma^{BC})). \end{aligned} \quad (10)$$

The relative entropy of entanglement [4] for  $\rho^{BC}$  is defined as

$$E_r(\rho^{BC}) = \min_{\sigma^{BC}_{\text{sep}}} S(\rho^{BC}||\sigma^{BC}) \quad (11)$$

where we've minimized over separable  $\sigma^{BC}$ . From now on, let us choose  $\sigma^{BC}$  to be the separable density operator that does this minimization for Bob and Claire's state  $\rho^{BC}$ . Then we obtain

$$\sum_k p_k S(\rho_k^{BC}||\sigma^{BC}) - E_r(\rho^{BC}) = \sum_k p_k S(\rho_k^{BC}||\rho^{BC}), \quad (12)$$

which means that

$$\begin{aligned} & \sum_k p_k E_r(\rho_k^{BC}) - E_r(\rho^{BC}) \\ & \leq \sum_k p_k S(\rho_k^{BC}||\rho^{BC}) \\ & = S(\rho^{BC}) - \sum_k p_k S(\rho_k^{BC}) \\ & = S(\rho^A) - \sum_k p_k S(\rho_k^A); \end{aligned} \quad (13)$$

the last step is valid since  $\rho^{ABC}$  and  $\rho_k^{ABC}$  are pure.

Now we consider a further step in which Alice communicates to Bob and Claire the results of her measurement and then Bob and Claire perform unitary rotations on their states with operators which depend on the outcome; we also allow Alice to perform rotations on her state which depend on the outcome of the measurement. We will denote by  $\tilde{\rho}_k^{BC}$  and  $\tilde{\rho}_k^A$  the states after the transformations i.e.

$$\begin{aligned} \rho_k^{BC} & \mapsto \tilde{\rho}_k^{BC} = U_k^B \otimes U_k^C \rho_k^{BC} \otimes (U_k^C)^\dagger; \\ \rho_k^A & \mapsto \tilde{\rho}_k^A = U_k^A \rho_k^A (U_k^A)^\dagger. \end{aligned} \quad (14)$$

These transformations do not change  $S(\rho_k^A)$ .  $E_r(\rho_k^{BC})$  is also left unchanged by these transformations since the set of separable states is invariant under local unitary transformations (i.e if  $\sigma^{BC}$  is separable, then so is  $U \otimes V \sigma^{BC} U^\dagger \otimes V^\dagger$ ) so that we find

$$\begin{aligned} & \sum_k p_k E_r(\tilde{\rho}_k^{BC}) - E_r(\rho^{BC}) \\ & \leq S(\rho^A) - \sum_k p_k S(\tilde{\rho}_k^A); \end{aligned} \quad (15)$$

Thus, the average increase in  $E_r$  for the Bob-Claire system is no greater than the average decrease in the entropy

of Alice's system (and thus Alice's entanglement with the joint Bob-Claire system).

Inequality (15) is also true in a step of an extended protocol in which Bob performs a measurement communicates to Alice and Claire, and then all three parties perform unitary transformations dependent on the outcome of the measurement. For again we consider a particular stage in the protocol and a particular branch in which the density matrix of the system is  $\rho^{ABC}$  (recall that the state is pure). Let Bob perform a measurement of an operator with spectral projectors  $P_k$ . Thus, as before, if the outcome  $k$  is obtained, we denote the state of the system  $\rho_k^{ABC}$  and  $p_k$  is the probability that the outcome  $k$  is obtained. Alice's reduced state after the measurement is

$$\rho_k^A = \text{Tr}_{BC}(\rho_k^{ABC}). \quad (16)$$

Before Bob communicates to her, her average state is

$$\rho^A = \sum_k p_k \rho_k^A. \quad (17)$$

Thus the convexity of entropy shows that

$$S(\rho^A) \geq \sum_k p_k S(\rho_k^A). \quad (18)$$

Now Bob communicates the outcome of his measurement and Alice performs a unitary transformation which depends on this outcome:

$$\rho_k^A \mapsto \tilde{\rho}_k^A = U_k^A \rho_k^A (U_k^A)^\dagger. \quad (19)$$

These transformations do not change  $S(\rho_k^A)$ . Thus, during this step of the protocol,

$$S(\rho^A) \geq \sum_k p_k S(\tilde{\rho}_k^A). \quad (20)$$

It is a key property of the relative entropy of entanglement that it does not increase under local operations and classical communication (see for example [4]) so that

$$E_r(\rho^{BC}) \geq \sum_k p_k E_r(\tilde{\rho}_k^{BC}). \quad (21)$$

Thus (20) and (21) imply that (15) is true for any step in the protocol.

If we consider an extended protocol in which Alice, Bob and Claire perform many rounds of local measurement, classical communication and unitary transformations, we may apply the above inequality to each round for each branch. We can then deduce that for any local protocol, the average increase in  $E_r$  for the Bob-Claire system is no greater than the average decrease in the Alice's entanglement with the joint Bob-Claire system. i.e. we may write

$$\begin{aligned} \langle E_r(BC) \rangle_{\text{final}} - E_r(BC)_{\text{initial}} \\ \leq S(A)_{\text{initial}} - \langle S(A) \rangle_{\text{final}}. \end{aligned} \quad (22)$$

The general question we are interested in is reversible procedures for converting a given state to some specified states. For a reversible process we know that average entropy cannot change since entropy can only stay constant or decrease under local operations. Thus in a reversible process the right-hand-side of (22) is zero and so in such a process the average relative entropy of entanglement of the Bob-Claire system must be constant.

This form of the result makes it very easy to show that it is not possible to convert GHZ states reversibly into singlets between Alice and Bob, Bob and Claire, and Alice and Claire. This is because the relative entropy of entanglement of GHZ states between Bob and Claire is zero, but clearly singlets between Bob and Claire have non-zero relative entropy of entanglement.

The results above lead us to our second main point namely to make quantitative statements about multi-party entanglement. Having established that GHZs and singlets are not interconvertible it is natural to ask, following [2] whether three party pure states can be transformed reversibly into singlets *and* three-party GHZ's. That is, perhaps singlets and three-party GHZ's constitute the irreducible types of entanglement into which any three party entanglement can be transformed reversibly. We do not know whether this is possible in general. However for those states  $\phi^{ABC}$  for which it is possible, our results easily show how many singlets and GHZ's can be extracted. This is since the one-party entropies and the relative entropy of entanglement of the reduced two-party density matrices are conserved as we have shown. Thus if  $g$  is the number of GHZ's that can be extracted per individual copy of  $\phi^{ABC}$ , and if  $s_{AB}$ ,  $s_{BC}$ , and  $s_{AC}$  are the number of singlets, then

$$\begin{aligned} S_A(\phi^{ABC}) &= g + s_{AB} + s_{AC} \\ S_B(\phi^{ABC}) &= g + s_{AB} + s_{BC} \\ S_C(\phi^{ABC}) &= g + s_{AC} + s_{BC} \\ E_r(\rho^{AB}) &= s_{AB} \\ E_r(\rho^{BC}) &= s_{BC} \\ E_r(\rho^{AC}) &= s_{AC}, \end{aligned} \quad (23)$$

where  $\rho^{AB}$  etc. are the reduced density matrices of  $\phi^{ABC}$ . i.e. the number of singlets between each pair  $AB$ ,  $BC$ ,  $AC$  that can be extracted per state asymptotically is equal to the relative entropy of entanglement of the reduced density matrices. We note that (23) shows that for states which are convertible into GHZ's and singlets, there are relations between the one-party entropies and relative entropies.

One interesting case is the state

$$\phi_1 = \alpha|000\rangle + \beta|111\rangle. \quad (24)$$

The conservation laws above suggest that the number of GHZ's that can be extracted is equal to

$$H(\alpha^2) = -\alpha^2 \log \alpha^2 - \beta^2 \log \beta^2. \quad (25)$$

Indeed a simple extension of the standard purification protocol [1] shows that this is indeed possible.

A second interesting case is

$$\phi_2 = \alpha|0\rangle\Psi_+ + \beta|1\rangle\Psi_-, \quad (26)$$

where  $\Psi_{\pm} = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ . It is very tempting to think that this state can indeed be transformed asymptotically into GHZ's and singlets, and the arguments above lead to a conjecture for the numbers of these states which can be extracted, namely  $H(\alpha^2)$  GHZ's and  $1 - H(\alpha^2)$  singlets between Bob and Claire, per copy of  $\phi_2$ . At present there is no protocol known to perform this transformation.

Our argument can be extended to situations in which multisystem entanglement is shared among more than three separated parties. Suppose  $n$  parties share  $n$  quantum systems in a joint entangled state. As long as  $n \geq 3$ , we can partition the  $n$  parties into three non-empty groups, which will play the roles of Alice, Bob and Claire. Local operations by any of the  $n$  parties will necessarily be local operations with respect to the Alice/Bob/Claire partition. Any increase in the relative entropy of entanglement between the Bob and Claire groups due to operations by the Alice group will necessarily involve an irreversible reduction in the entropy of the Alice group, and thus a reduction in the entanglement of the Alice group with the others.

Any entangled pure state of  $k$  parties must show bipartite entanglement between some subset of the  $k$  parties and the complementary subset. This fact allows us to draw conclusions about the reversible transformations of many-particle entanglement. For instance, a GHZ state shared among  $n$  parties has the property that for  $k < n$ , no  $k$  parties are entangled among themselves. Thus,  $n$ -party GHZ's cannot be reversibly transformed into any combination of  $k$ -party entangled pure states, for all  $k < n$ .

Finally we point out that in our derivation of (22), we had in mind the definition of relative entropy of entanglement given by

$$E_r(\rho^{BC}) = \min_{\sigma^{BC}_{\text{sep}}} S(\rho^{BC} || \sigma^{BC}). \quad (27)$$

That is, the set of states  $\Sigma$  over which we minimize is the set of separable states. However, the only property of that set necessary for the proof was the *invariance* of that set under local transformations, that is unitary transformations and measurements (in particular [4] the invariance under measurements enters in the derivation of (21)). For two parties, a natural such set is the set of separable states. For larger numbers of parties, more

general choices are possible (see for example [4]), and can provide new measures of multisystem entanglement.

For example for four parties, we can take the states in  $\Sigma$  to be mixtures of pure states of the form  $|\psi^{ABC}\rangle \otimes |\phi^D\rangle$ , or we could take mixtures of these states and similar states with  $ABCD$  permuted. Or we could take  $\Sigma$  to be mixtures of pure states of the form  $|\psi^{AB}\rangle \otimes |\phi^{CD}\rangle$  and permutations etc. In any of these cases the set  $\Sigma$  is invariant under local transformations. We can therefore use it to define  $E_\Sigma(\rho^{ABCD})$ , the relative entropy “distance” of a state  $\rho^{ABCD}$  from the set  $\Sigma$ . Similar reasoning to that given earlier allows us to derive inequalities similar to (15):

$$\begin{aligned} \sum_k p_k E_\Sigma(\tilde{\rho}_k^{BCDE}) - E_\Sigma(\rho^{BCDE}) \\ \leq S(\rho^A) - \sum_k p_k S(\tilde{\rho}_k^A); \end{aligned} \quad (28)$$

More generally for any  $n$  we can consider the set  $\Sigma$  to be states which are mixtures of pure states with any given partitioning of all the parties. A hierarchy of entangle-

ment measures emerges, each member of which must be conserved in reversible transformations.

We are very grateful to Serge Massar for discussions and for pointing out a mistake in an early version of the manuscript. This work was made possible by the Isaac Newton Institute programme on “Complexity, Computation and the Physics of Information” (1999), partly supported by the European Science Foundation. One of us (BS) also acknowledges the support of a Rosenbaum Fellowship during this programme.

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